HERON TRILATERALS
BHASKARA EQUATIONS
CONTINUED FRACTIONS
1

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Contents

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0. Wandering in the Wonderful world of Numbers

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Last winter I was on a wintering-holiday in India. And if you can point your antennas on it, you will find history of mathematics everywhere. At the observatories in Jaipur and Varanasi, at a temple with an ancient magic square in Khajuraho, you have the feeling of walking in the footsteps of historical mathematicians. After all, they were in many ways arithmatic pioneers in India. In the time of the European mathematicians Euler, de Fermat, Mersenne, and Huygens, all kinds of arithmetic properties were rediscovered. Some have been proven, others are still sources of research and conjecture.
One of the areas of these ancient times concerns Heron triangles, triangles with sides and area equal to integers. Heron is the Greek hero of the story and Bhaskara and Brahmagupta are the heroes of India.
The question that came to my mind is, "Would there be a tetrahedron with four Heron triangles and integer volume?" [9]. And then the next question was: "Are there enough triangles with sides that fit together to make such a tetrahedron at all?"
In searching this matter I came across the concept of Heron trilaterals. (See figure 3). I call the part of the figure of the side AB the Heron skeleton. (See figure 4). The tangent point C_0 of the inner circle lies on the side AB. I have made a program with SageMath. It needs input $c = AB$ and $x = AC_0$. It gives as ouput the possible Heron triangles, which can be constructed on the Heron skeleton (c, x).
I looked for Heron triangles with a side of given length and I found a wonderful world of numbers. The world of positive integers, of the rational numbers, of the quadratic numbers, of Bhaskara's equations, once erroneously called Pell's equations, of the (finite and periodic) continued fractions and their convergents.

Section 1 provides an introduction to Heron trilaterals. Because I will not be concerned with degeneracies, I will only use the set $\mathbb{N} = \mathbb{Z}^+$ for the positive integers. After all, 0 has ever been added. Section 2 follows the construction of the heroncx program in SageMath. The listing of the program is included in Appendix A. Section 3 contains some findings when playing with heroncx. It contains enough questions to expand this article with some chapters on continued fractions and solutions of the Bhaskara equation $x^2 - dy^2 = c^2$ from sections 2.3 and 2.4. and who knows what else.

Hein van Winkel.

1. Heron trilaterals

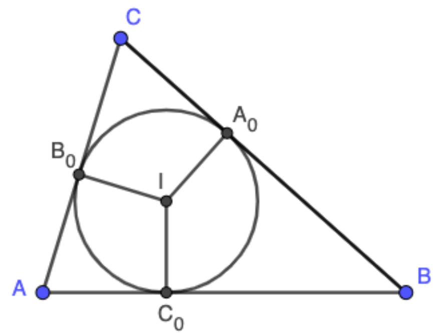

Figure 1. Heron triangle

Notations. The sidelengths of BC, AC, AB are respectively a, b, c. The sizes of the interior angles of $\triangle ABC$ are $\alpha = \angle A, \beta = \angle B, \gamma = \angle C$. The tangentpoints of the incircle with the sides a, b, c of the triangle are respectively A_0, B_0, C_0. The lenghts of the tangents from the vertices to these tangentpoints are $AB_0 = AC_0 = s_a, BA_0 = BC_0 = s_b, CA_0 = CB_0 = s_c$. The sum of these six tangents is the perimeter of the triangle. The half perimeter is the first letter s of semi-perimeter. The incircle has midpoint I and radius r. The letter F is used for the size of the area of the triangle.

Definition 1.1. $\triangle ABC$ is a Heron triangle if $a, b, c, F \in \mathbb{N}$.

Some basic properties without proof:

Proposition 1.2. $s_a = s - s_b - s_c = s - a, s_b = s - b, s_c = s - c$

Proposition 1.3. $s = s_a + s_b + s_c$

Proposition 1.4 (Heron's formula). $F = \sqrt{s \cdot s_a \cdot s_b \cdot s_c}$

Proposition 1.5. $F = r \cdot s$

Proposition 1.6. $r^2 s = s_a \cdot s_b \cdot s_c \Leftrightarrow r^2(s_a + s_b + s_c) = s_a \cdot s_b \cdot s_c$

Proposition 1.7. $\frac{s_a}{r} + \frac{s_b}{r} + \frac{s_c}{r} = \frac{s_a}{r} \cdot \frac{s_b}{r} \cdot \frac{s_c}{r}$

Proposition 1.8. In a Heron tirangle each off the $a, b, c, s_a, s_b, s_c, s, F \in \mathbb{N}$ and $r \in \mathbb{Q}$.

Adding the excircle with centre I_c at the side c of $\triangle ABC$ gives figure 2. The tangent points with the sides a, b, c or their extentions are respectively A_c, B_c, C_c, and the exradius $r_c = I_c C_c$.

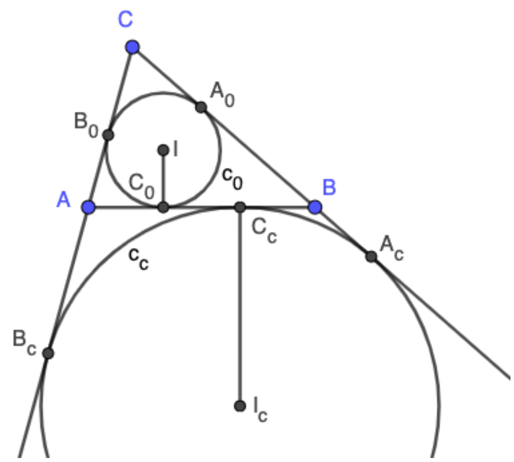

FIGURE 2. Excircle I_c

Another set of basic properties (some without proof):

Proposition 1.9. $CA_c = s$.

Proof: $CA_c + CB_c = 2 * CA_c = a + b + c = 2s \Rightarrow CA_c = s$.

Proposition 1.10. $AB_c = AC_c = s - b = s_b, BA_c = BC_c = s_a$

Proposition 1.11. $F = r_c s_c$
Proof: $\tan(\frac{\gamma}{2}) = \frac{r_c}{s} = \frac{r}{s_c} \Rightarrow r_c s_c = rs = F$.

Proposition 1.12. $rr_c = s_a s_b$
Proof : $AI \perp AI_c \Rightarrow \frac{s_b}{r_c} = \frac{r}{s_a} \Leftrightarrow rr_c = s_a s_b$

Proposition 1.13. $rr_a r_b r_c = F^2$.
Proof : According to 1.11 is $rr_a r_b r_c = \frac{F}{s}\frac{F}{s_a}\frac{F}{s_b}\frac{F}{s_c}$ and according to Heron's formula is $\sqrt{s \cdot s_a \cdot s_b \cdot s_c} = F$. Then it follows that $rr_a r_b r_c = \frac{F}{s}\frac{F}{s_a}\frac{F}{s_b}\frac{F}{s_c} = F^4/F^2 = F^2$ is.

Proposition 1.14. $F = \frac{rr_a r_b r_c}{rs} = \frac{r_a r_b r_c}{s}$

Proposition 1.15. $r_a r_b + r_b r_c + r_c r_a = s^2$
Proof : $r_a r_b + r_b r_c + r_c r_a = r_a r_b r_c(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}) = sF(\frac{s_a}{F} + \frac{s_b}{F} + \frac{s_c}{F}) = s^2$

Definition 1.16. A Heron trilateral is a configuration of three lines a, b and c, intersecting each other in three different points A, B, C such that $\triangle ABC$ is a Heron triangle. If there is no confusion the letters a, b, c are used as the lines and as the lenght of the sides of the triangle. (See figure 3)

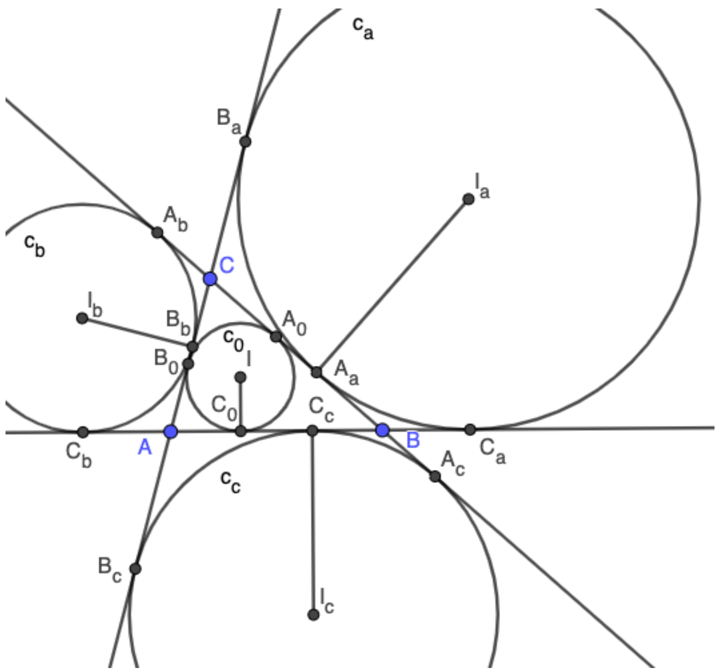

FIGURE 3. Heron trilateral

Notations. The incircle of $\triangle ABC$ with center I is tangent at the points A_0, B_0, C_0 on the side a, b, c respectively. The excircle, tangent at side a, with center I_a is tangent in the points A_a, B_a, C_a at the side a, b, c respectively. The radius of this circle is r_a.
The length of the tangent from C to the excircle with center I_c is defined to be the negative number $s'_c = -s$ and the length of the tangents from A and B at circle I_c are $s'_a = s_b$ and $s'_b = s_a$ respectively. By definition is $s' = s'_a + s'_b + s'_c$.

Propositon 1.17. $s' = -s_c$ and $r' = -r$.
Proof : $s' = s'_a + s'_b + s'_c = s_b + s_a - s = -s_c$.
From $F = r \cdot s = r' \cdot s'$ follows $r' = \frac{r \cdot s}{s'} = \frac{r \cdot s}{-s_c} = -r_c$.

Proposition 1.18. With the notations of s' and s'_a, heron's formel is still valid.
Proof : $F = \sqrt{s' \cdot s'_b \cdot s'_a \cdot s'_c} = \sqrt{-s_c \cdot s_a \cdot s_b \cdot -s} = \sqrt{s \cdot s_a \cdot s_b \cdot s_c}$

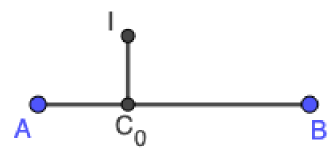

FIGURE 4. Heron skeleton

Definition 1.19. After removing quite a bit from the figures in this section a so-called Heron skeleton remains. It consists of the side AB with length c of $\triangle ABC$ and the radius IC_0 of the incircle with their tangent point. The length of AC_0, BC_0, C_0I is s_a, s_b, r respectively as in the foregoing. The length s_a and s_b are in $\mathbb{N}$. The length of r is in $\mathbb{Q}$. In section 2 follows an investigation of the possible values of r such that the skeleton can be extended to an heron trilateral.

Definition 1.20. Let M be the center of the line segment A_0A_c. Then by definition $CM = t_c$.

Proposition 1.21. $t_c = s_c + c/2 = (s + s_c)/2$.
Proof: $CM = t_c = s_c + \frac{1}{2}(s - s_c) = \frac{1}{2}(s + s_c) = (s_a + s_b + 2s_c)/2 = (c + 2s_c)/2 = s_c + c/2$.

Proposition 1.22. If c is even then $t_c \in \mathbb{N}$. If $c =$ odd then $t_c + \frac{1}{2} \in \mathbb{N}$
Proof. This follows immediately from 1.21.

Theorem 1.23 If (t_c, s_a, s_b) describes the Heron trilateral (a, b, c) then $(-t_c, s_a, s_b)$ describes the same Heron trilateral $H(a, b, c)$.
Proof.
$s_c = t_c - c/2$ by (1.21) and
$s = s_c + c = s_c + c/2 + c/2 = t_c + c/2$.
For tangents at the excircle I_c holds:
$s'_a = s_b$
$s'_b = s_a$
$s'_c = -s = -(t_c + c/2) = (-t_c) - c/2$
$s' = -s_c = -(t_c - c/2) = (-t_c) + c/2$.
And this proves that $(-t_c, s_a, s_b)$ describes the same Heron trilateral.

Proposition 1.24. The inradius of an heron-skeleton is less than $\sqrt{s_a s_b}$.
Proof. Let the skeleton in figure 5 have $CD = \sqrt{AC \cdot CB} = \sqrt{s_a s_b}$. Then $AD \perp BD$. Then

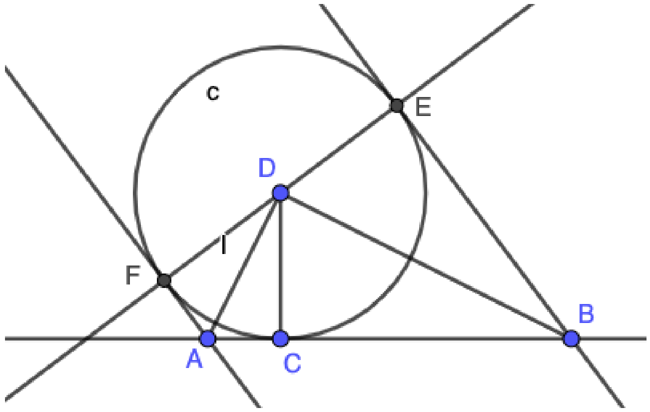

FIGURE 5. limit figure

the tangentpoints E and F are collinear with D and the tangents AF and BE are parallel and there is no triangle. When $CD < \sqrt{s_a s_b}$ the tangents intersect each other in a point at the same side of AB as point D does and the circle is the incircle. When $CD > \sqrt{s_a s_b}$ the tangents intersect each other in a point at the other side of AB as point D does and the circle is the excircle at side AB of the triangle.
REMARK : By proposition 1.12 is $rr_c = s_a s_b$.

2. Incircle radius of heron trilaterals

This section explores the possible values of the inradius r such that the skeleton of figure 4 is a Heron skeleton. The solution to the problem is highly dependent on the values of $p = s_a s_b$ and $c = s_a + s_b$.
Use is made of some of the statements **(C1 - C6)** formulated in the article by Keith Conrad [3], [4] :

- **Theorem C.1** ([3]Theorem 4.1). If $X^2 - dY^2 = 1$ has solutions (x, y) and (x', y') then the coefficients of $(x + y\sqrt{d})(x' + y'\sqrt{d})$ are also a solution.

- **Corollary C.2** ([3]Corollary 4.2).. If $X^2 - dY^2 = 1$ has a solution (x, y) then the coefficients of $(x + y\sqrt{d})^n$ are also a solution for all $n \in \mathbb{Z}$. In particular, this Pell equation has infinitely many solutions if it has a nontrivial solution.

- **Theorem C.3** ([3]Theorem 5.3).. Assume $x^2 - dy^2 = 1$ has a solution in positive integers and let (x_1, y_1) be such a solution where y_1 is minimal. Then all solutions to $x^2 - dy^2 = 1$ in integers are, op to sign, generated from (x_1, y_1) by taking powers of $x_1 + y_1\sqrt{d}$:
$$x + y\sqrt{d} = \pm(x_1 + y_1\sqrt{d})^n$$
for some $n \in \mathbb{Z}$ and some sign.

- **Theorem C.4** ([4]Theorem 2.3). (Lagrange, 1768). For any positive integer d that is not a square, the equation $x^2 - dy^2 = 1$ has a nontrivial solution.

- **Theorem C.5** ([4]Theorem 3.3).. Fix $u = a + b\sqrt{d}$ where $a^2 - db^2 = 1$ with a and b in $\mathbb{Z}^+$. For each $n \in \mathbb{Z} - \{0\}$, every solution of $x^2 - dy^2 = n$ is a Pell multiple of a solution (x, y) where
$$\mid x \mid \leq \sqrt{\mid n \mid u} \text{ and } \mid y \mid \leq \sqrt{\mid n \mid u}/\sqrt{d}$$
- **Corollary C.6** ([4]Corollary 3.4). . For any generalized Pell equation $x^2 - dy^2 = n$ with $n \neq 0$ there is a finite set of solutions such that every solution is a Pell multiple of one of these solutions.

Heron's formula $F^2 = s_a s_b s_c s = (s_a s_b) s_c (s_c + s_a + s_c =$
$$F^2 = p s_c (s_c + c) \tag{1}$$
Solving this equation depends on the values of p and c. The different solutions are described in the following subsections. Appendix A contains the program listing of the function heroncx with input respectively the length of AB and AC_0 in figure 4 and output the heron trilaterals in the format (inradius, a, b, c, area).

2.1. **SQE (p is a perfect square and c is even).** .
Let $p_1^2 = p$. From equation 1 follows $p_1^2 = p|F^2 \Rightarrow p_1|F \Rightarrow F = p_1 B$ with $B \in \mathbb{N}$
Let $c_0 = c/2 \Rightarrow t_c = s_c + c_0 (t_c \in \mathbb{N})$.
After substitutions and some math equation 1 becomes

$$\begin{aligned} ps_c(s_c + c) = F^2 &\Leftrightarrow p_1^2(s_c^2 + 2s_c c_0) = p_1^2 B^2 \\ &\Leftrightarrow s_c^2 + 2s_c c_0 = B^2 \\ &\Leftrightarrow (s_c + c_0)^2 = B^2 + c_0^2 \end{aligned}$$

$$t_c^2 - B^2 = c_0^2 \tag{2}$$

Solving equation (2) in $\mathbb{N}$ gives:

$$(t_c + B)(t_c - B) = c_0^2 \Leftrightarrow t_c + B = d_1 \wedge t_c - B = d_2$$
$$t_c = \frac{d_1 + d_2}{2} = \frac{d_1^2 + c_0^2}{2d_1} \wedge B = \frac{d_1 - d_2}{2} = \frac{d_1^2 - c_0^2}{2d_1}$$

with $(d_1 d_2 = c_0^2)$, $d_1 > d_2 > 0$ and d_1 and d_2 must have the same parity, because their sum and difference must be divisible by 2.
The solution (t_c, B) gives $s_c + c_0 = t_c$ and $F = p_1 B$ and

$$\begin{cases} s_c = t_c - c_0 & = \frac{d_1^2 + c_0^2}{2d_1} - c_0 & = \frac{(d_1 - c_0)^2}{2d_1} \\ s = t_c + c_0 & = \frac{d_1^2 + c_0^2}{2d_1} + c_0 & = \frac{(d_1 + c_0)^2}{2d_1} \\ F = p_1 B & = p_1 \cdot \frac{d_1^2 - c_0^2}{2d_1} & \\ r = \frac{p_1 B}{t_c + c_0} & = p_1 \cdot \frac{d_1^2 - c_0^2}{2d_1} \cdot \frac{2d_1}{(d_1 + c_0)^2} & = p_1 \cdot \frac{d_1 - c_0}{d_1 + c_0} \end{cases} \tag{3}$$

Example 2.1 sqe with $c = 10, p = 9$, equivalent $s_a = 1, s_b = 9$.

$$\begin{aligned} ps_c(s_c + c) = F^2 &\Leftrightarrow 9(s_c^2 + 10s_c) = 9B^2 \\ &\Leftrightarrow t_c^2 - B^2 = 25 \end{aligned}$$

d_1	d_2	t_c	B	s_c	s	a	b	c	F	r	r_a	r_b	r_c
25	1	13	12	8	18	17	9	10	36	2	36	4	$\frac{9}{2}$

2.2. **SQO (p is a perfect square and c is odd).** .
Let $p_1^2 = p$. From equation 1 follows $p_1^2 = p|F^2 \Rightarrow p_1|F \Rightarrow F = p_1 B$ with $B \in \mathbb{N}$
$t_c = s_c + c/2 \Rightarrow 2t_c = 2s_c + c$ with $2t_c \in \mathbb{N} \wedge t_c \notin \mathbb{N}$
After substitutions and some math equation 1 becomes

$$\begin{aligned} ps_c(s_c + c) = F^2 &\Leftrightarrow p_1^2(s_c^2 + s_c c) = p_1^2 B^2 \\ &\Leftrightarrow s_c^2 + c \cdot s_c = B^2 \Leftrightarrow 4s_c^2 + 4c \cdot s_c = 4B^2 \\ &\Leftrightarrow 4s_c^2 + 4c \cdot s_c + c^2 = 4B^2 + c^2 \Leftrightarrow (2t_c)^2 = (2B)^2 + c^2 \end{aligned}$$

With $2t_c = X$ odd and $2B = Y$ even

$$\Leftrightarrow X^2 - Y^2 = c^2 \tag{4}$$

Solving equation (4) in $\mathbb{N}$ gives:

$$(X+Y)(X-Y) = c^2 \Leftrightarrow X+Y = d_1 \wedge X-Y = d_2$$
$$X = \frac{d_1+d_2}{2} \wedge Y = \frac{d_1-d_2}{2}$$

with $d_1 d_2 = c^2$, $d_1 > d_2 > 0$. Because $d_1 d_2 = c^2$ is odd, d_1 and d_2 are both odd and X, Y must be in $\mathbb{N}$..
The solution $(X,Y) = (2t_c, 2B)$ gives
$t_c = X/2 = \frac{d_1+d_2}{4} = \frac{d_1+\frac{c^2}{d_1}}{4} = \frac{d_1^2+c^2}{4d_1}$ and
$B = \frac{Y}{2} = \frac{d_1-d_2}{4} = \frac{d_1-\frac{c^2}{d_1}}{4} = \frac{d_1^2-c^2}{4d_1}$.
The solution (t_c, B) gives :

$$\begin{cases} s_c = t_c - \frac{c}{2} & = \frac{d_1^2+c^2}{4d_1} - \frac{c}{2} & = \frac{(d_1-c)^2}{4d_1} \\ s = t_c + \frac{c}{2} & = \frac{d_1^2+c^2}{4d_1} + \frac{c}{2} & = \frac{(d_1+c)^2}{4d_1} \\ F = p_1 B & = p_1 \cdot \frac{d_1^2-c^2}{4d_1} & \\ r = p_1 \cdot \frac{d_1^2-c^2}{4d_1} \cdot \frac{4d_1}{(d_1+c)^2} & = p_1 \cdot \frac{d_1-c}{d_1+c} & \end{cases} \tag{5}$$

Example 2.2 sqo with $c = 13, p = 36$, equivalent $s_a = 4, s_b = 9$.

$$ps_c(s_c+c) = F^2 \Leftrightarrow 36(s_c^2 + 13s_c) = 36B^2$$
$$\Leftrightarrow (2s_c+13)^2 = 4B^2 + 169$$
$$\Leftrightarrow (2t_c)^2 - (2B)^2 = 169$$

d_1	d_2	$2t_c$	$2B$	s_c	s	a	b	c	F	r	r_a	r_b	r_c
169	1	85	84	36	49	45	40	13	252	$\frac{36}{7}$	63	28	7

2.3. **NQE (p is not a perfect square and c is even).** .
Let p_0 be the square-free part of $p = p_0 p_1^2 \Rightarrow p_0 | F^2 \Rightarrow p_0 | F \Rightarrow F = p_0 U$ with $U \in \mathbb{N}$
Let $c_0 = c/2$ and $t_c = s_c + c_0$.
After substitutions equation 1 becomes:

$$ps_c(s_c+c) = F^2 \Leftrightarrow p_0 p_1^2 (s_c^2 + 2s_c c_0) = p_0^2 U^2$$
$$\Leftrightarrow p_1^2 (s_c^2 + 2s_c c_0) = p_0 U^2$$

Here is $p_1^2|p_0U^2$ with p_0 square-free. If q is a prime factor of $\gcd(p_0,p_1)$ then p_1^2 must have an even number of factors q en thus $p_1^2|U^2 \Rightarrow U=p_1B$ with $B\in\mathbb{N}$. And the equation becomes now:

$$\begin{aligned} ps_c(s_c+c)=F^2 &\Leftrightarrow p_1^2(s_c^2+2s_cc_0)=p_0p_1^2B^2\\ &\Leftrightarrow s_c^2+2s_cc_0=p_0B^2\\ &\Leftrightarrow s_c^2+2s_cc_0+c_0^2=p_0B^2+c_0^2\\ &\Leftrightarrow t_c^2-p_0B^2=c_0^2\end{aligned}$$

Then (t_c,B) is a solution to the Bhaskara-Pell-equation

$$\Leftrightarrow X^2-p_0Y^2=c_0^2 \tag{6}$$

This equation has for all the values of p_0 and c_0 in the context of this article one or more series of solutions. (See C.6 on page 8). Let $\alpha=\alpha_0+\alpha_1\sqrt{p_0}=(\alpha_0,\alpha_1)$ be the solution to $X^2-p_0Y^2=1$ with $\alpha_0,\alpha_1\in\mathbb{N}$ and α_1 minimal. Let $\beta=\beta_0+\beta_1\sqrt{p_0}=(\beta_0,\beta_1)$ be a solution to $X^2-p_0Y^2=c_0^2$. There is allways one such β, namely $\beta=c_0$. This gives the series solutions $\pm\beta\alpha^i, i\in\mathbb{Z}$. There exist a finite number of such series. Restriction to $X,Y>0$ does not result in a loss of Heron trilaterals according theorem 1.23.

The solution $(t_c,B)=(X,Y)$ to equation 6 gives :

$$\begin{cases} s_c=t_c-\frac{c}{2} & =\frac{2t_c-c}{2} & =\frac{2X-c}{2}\\ s=t_c+\frac{c}{2} & =\frac{2t_c+c}{2} & =\frac{2X+c}{2}\\ F=p_0U & =p_0p_1B & =p_0p_1Y\\ r=\frac{F}{s} & =\frac{2p_op_1B}{2t_c+c} & =\frac{2p_op_1Y}{2X+c}\end{cases} \tag{7}$$

Example 2.3 nqe with $c=10$ and $p=4\cdot6=24$ and so $p_0=6,p_1=2,s_a=4,s_b=6$.

$$\begin{aligned} ps_c(s_c+c)=F^2 &\Leftrightarrow 24(s_c^2+10s_c)=F^2\\ &\Leftrightarrow 6\cdot2^2(s_c^2+10s_c)=6^2U^2\\ &\Leftrightarrow 2^2(s_c^2+10s_c)=6U^2 \Leftrightarrow 2^2(s_c^2+10s_c)=6\cdot(2B)^2\\ &\Leftrightarrow s_c^2+10s_c=6B^2 \Leftrightarrow (s_c^2+5)^2=6B^2+25\\ &\Leftrightarrow X^2-6Y^2=25\end{aligned}$$

$5+2\sqrt{6}$ is the 'smallest' solution to $X^2-6Y^2=1$.
$(5,0)$ is solution to $X^2-6Y^2=25$
One of the series of solutions is $\pm5(5+2\sqrt{6})^i$, starting with $\pm(25+10\sqrt{6}),\pm(245+100\sqrt{6}),\cdots,\pm(25-10\sqrt{6}),\cdots$. From this series follows with positive X,Y:

$X=t_c$	$Y=B$	s_c	s	a	b	c	F	r	r_a	r_b	r_c
25	10	20	30	26	24	10	120	4	230	20	6
245	100	240	250	1200	246	244	10	$\frac{24}{5}$	300	200	5

2.4. **NQO (p is not a perfect square and c is odd).** .
Let p_0 be the square-free part of $p = p_0 p_1^2 \Rightarrow p_0|F^2 \Rightarrow p_0|F \Rightarrow F = p_0 U$ with $U \in \mathbb{N}$
So the equation 1 becomes:

$$\begin{aligned} ps_c(s_c + c) = F^2 &\Leftrightarrow p_0 p_1^2 (s_c^2 + s_c c) = p_0^2 U^2 \\ &\Leftrightarrow p_1^2 (s_c^2 + s_c c) = p_0 U^2 \end{aligned}$$

Here is $p_1^2 | p_0 U^2$ with p_0 square-free. If q is a prime factor of $\gcd(p_0, p_1)$ then p_1^2 must have an even number of factors q en thus $p_1^2 | U^2 \Rightarrow U = p_1 B$ with $B \in \mathbb{N}$.
c is odd $\Rightarrow 2t_c = 2s_c + c$ is odd too.
And the equation 1 becomes:

$$\begin{aligned} ps_c(s_c + c) = F^2 &\Leftrightarrow p_1^2 (s_c^2 + s_c c) = p_0 p_1^2 B^2 \\ &\Leftrightarrow s_c^2 + s_c c = p_0 B^2 \\ &\Leftrightarrow 4s_c^2 + 4s_c c = 4p_0 B^2 \\ &\Leftrightarrow 4s_c^2 + 4s_c c + c^2 = 4p_0 B^2 + c^2 \Leftrightarrow (2t_c)^2 = p_0 (2B)^2 + c^2 \\ &\Leftrightarrow (2t_c)^2 - p_2 B^2 = c^2 \end{aligned}$$

With $p_2 = 4p_0$.

Then $(2t_c, B)$ is a solution of the Bhaskara-Pell-equation

$$X^2 - p_2 Y^2 = c^2 \tag{8}$$

This equation has for all the values of p_2 and c in the context of this article one or more series of solutions. (See C.6 on page 8). Let $\alpha = \alpha_0 + \alpha_1 \sqrt{p_0} = (\alpha_0, \alpha_1)$ be the solution to $X^2 - p_2 Y^2 = 1$ with $\alpha_0, \alpha_1 \in \mathbb{N}$ and α_1 minimal. Let $\beta = \beta_0 + \beta_1 \sqrt{p_0} = (\beta_0, \beta_1)$ be a solution to $X^2 - p_0 Y^2 = c^2$. There is allways one such β, namely $\beta = c$. This gives the series solutions $\pm \beta \alpha^i, i \in \mathbb{Z}$. There exist an finite number of such series.
The solution $(X, Y) = (2t_c, B)$ to equation 8 gives :
The solution (t_c, B) gives :

$$\begin{cases} s_c = t_c - \frac{c}{2} & = \frac{2t_c - c}{2} & = \frac{X - c}{2} \\ s = t_c + \frac{c}{2} & = \frac{2t_c + c}{2} & = \frac{X + c}{2} \\ F = p_0 U & = p_0 p_1 B & = p_0 p_1 Y \\ r = \frac{F}{s} & = \frac{2p_o p_1 B}{2t_c + c} & = \frac{2p_o p_1 Y}{X + c} \end{cases} \tag{9}$$

Example 2.4a nqo with $c = 9$ and $p = 3 \cdot 6 = 18$ and so $s_a = 3, s_b = 6, p_0 = 2, p_1 = 3$.

$$\begin{aligned} ps_c(s_c + c) = F^2 &\Leftrightarrow 18(s_c^2 + 9s_c) = F^2 \\ &\Leftrightarrow 2 \cdot 3^2 (s_c^2 + 9s_c) = 2^2 U^2 \\ &\Leftrightarrow 3^2 (s_c^2 + 9s_c) = 2U^2 \Leftrightarrow 3^2 (s_c^2 + 9s_c) = 2 \cdot 3^2 B^2 \\ &\Leftrightarrow s_c^2 + 9s_c = 2B^2 \Leftrightarrow 4s_c^2 + 4 \cdot 9s_c + 81 = 8B^2 + 81 \\ &\Leftrightarrow (2s_c + 9)^2 - 8B^2 = 81 \Leftrightarrow (2t_c)^2 - 8B^2 = 81 \end{aligned}$$

$3+\sqrt{8}$ is the 'smalllest' solution to $X^2-8Y^2=1$.
9 is a solution to $X^2-8Y^2=81$.
One of the series solutions is $\pm 9(3+\sqrt{8})^i$, starting with $27+9\sqrt{8}, 153+54\sqrt{8}, \cdots, -9(3-\sqrt{8}) = -27+9\sqrt{8}, \cdots$.
We are interested in solutions with positive B, so we get:

$X=2t_c$	$Y=B$	s_a	s_b	s_c	s	F	a	b	c	r	r_a	r_b	r_c
27	9	3	6	9	18	54	15	12	9	3	18	9	6
153	54	3	6	72	81	324	78	75	9	4	108	54	$\frac{9}{2}$

Example 2.4b nqo with $c=11$ and $p=3\cdot 8=24$ and so $s_a=3, s_b=8, p_0=6, p_1=2$.

$$\begin{aligned} ps_c(s_c+c)=F^2 &\Leftrightarrow 24(s_c^2+11s_c)=F^2 \\ &\Leftrightarrow 6\cdot 2^2(s_c^2+11s_c)=6^2U^2 \\ &\Leftrightarrow 2^2(s_c^2+11s_c)=6U^2 \Leftrightarrow 2^2(s_c^2+11s_c)=6\cdot 2^2B^2 \\ &\Leftrightarrow s_c^2+11s_c=6B^2 \Leftrightarrow 4s_c^2+4\cdot 11s_c+121=24B^2+121 \\ &\Leftrightarrow (2s_c+11)^2-24B^2=121 \Leftrightarrow (2t_c)^2-24B^2=121 \end{aligned}$$

$5+\sqrt{24}$ is the 'smalllest' solution to $X^2-24Y^2=1$.
11 is a solution to $X^2-24Y^2=121$.
One of the series solutions is $\pm 11(5+\sqrt{24})^i$, starting with $55+11\sqrt{24}, 539+110\sqrt{24}, \cdots, -11(5-\sqrt{24}) = -55+11\sqrt{24}, \cdots$.
We are interested in solutions with positive B, so we get:

$X=2t_c$	$Y=B$	s_a	s_b	s_c	s	F	a	b	c	r	r_a	r_b	r_c
55	11	3	8	22	33	132	30	25	11	4	44	$\frac{33}{2}$	6
539	110	3	8	264	275	1320	272	267	11	$\frac{24}{5}$	440	165	5

2.5. **heron(c,x).** .
This subsection is a description of the heroncx(c,x) program with input a Heron skeleton and with output sometimes singular Heron trilaterals of some infinite series of Heron trilaterals.

INPUT-OUTPUT:

- INPUT

```
aantal = a
(table(heroncx(c,x)))
```

The last two lines of the program contain the input lines. Where c is the length of AB in the skeleton and $x = AC_0 = s_a$, the length of the tangent line from A to the incircle. The desired number of rows of the infinite series can be indicated with the number a.

- OUTPUT A table with a row for each Heron trilateral and a column for the values of r, a, b, c, F, which are the values of the inradius, the length of a, b, c and the area of the triangle. The first a triangles of the infinite series are noted.

STRUCTURE of the program (for the letters **A-U** see appendix A):

A: Here the program is divided into four parts, corresponding to (B-C)SQE(2.1), (D-E)SQO(2.2), (H-N)NQE(2.3) and (O-T)NQO(2.4) in section 2.
B: Initiation: definition/calculation of c_0, p_1 and the outputlist sol.
C: Divisors of c_0^2, solution of $t_c^2 - B^2 = c_0^2 \Leftrightarrow X^2 - Y^2 = c_0^2$ and formatting the ouput.
D: Initiation: definition/calculation of p_1 and the outputlist sol.
E: Divisors of c^2, solution of $(2t_c)^2 - (2B)^2 = c^2 \Leftrightarrow X^2 - Y^2 = c^2$ and formatting the ouput.
F: bhaskara(d). Program to find the 'smallest' solution of the Bhaskara (Pell) equation $x^2 - dy^2 = 1$ with input d.
G: bhaskara_g(d,g). Program to find the series of solutions of the Bhaskara equation $x^2 - dy^2 = n$ with input: (d, n). This part make use of F and an intern list sol of rows. The ouput is a solution as described in theorem C.3 in at the beginning of this section.
H-I: Start of NQE. Definition/calculation of c_0, p, p_0, p_1 and the outputlist sol.
J-M: Some series of solutions from G in the list solu can be the same series. In this part only one of each different series stays in solB.
N: Formatting the ouput.
O-P: Start of NQO. Definition/calculation of c_0, p, p_0, p_1, p_2 and the outputlist sol.
Q-S: See **J-M**.
T: Formatting the ouput.
U: Start of the program. Here the input values for number and heroncx must be entered.

3. Observations - Questions - Remarks

3.1. More obtuse-angled than acute-angled trilaterals for small area's. .

Nobody is perfect. I debugged the program and found trilaterals with some of the $a, b, c, F \notin \mathbb{N}$. Debugging was so intense at times that some trilaterals disappeared. I found table 1 of primitive heron triangles up to an area of 396 on wiki[7]. I decided to complete the debugging by checking if all the triangles from the table on input of 2 (isosceles triangles) or 3 (scalene triangles) skeletons were in the output. I added a column with I(soscele), A(cute), R(ight-angled) and O(btused-angled) triangles. I noticed that there were many more obtuse-angled triangles up to the area of 396.

3.2. $c = a + b$ with $a \cdot b$ is a perfect square. .

DEFINITION: A Heron skeleton with only a finite number of Heron trilaterals is called a *singular Heron skeleton*.

The corresponding Heron trilaterals have at least one of $s_a s_b$, $s_b s_c$ or $s_c s_a$ equal to a perfect square. This leads to the question: 'Which numbers c are the sum of two positive numbers a and b such that their product $a \cdot b$ is a perfect square?' See table 2 : , the singular Heron skeletons with $c < 150$ and p a perfect square. Remark : the first element $(2, 1)$ in this table is not a Heron skeleton

Let $n = 2^t * p_1^{a_1} * p_2^{a_2} * ... * p_r^{a_r} * q_1^{b_1} * q_2^{b_2} * ... * q_s^{b_s}$ for different p_i and q_j with $t \geq 0, a_i \geq 0, p_i \equiv_4 1, b_j \geq 0, q_j \equiv_4 3$ for $i = 1, \cdots, r; j = 1, \cdots, s$

Some sequences from OEIS:

A000404: Numbers that are the sum of 2 nonzero squares.
$n \in A000404 \Leftrightarrow (b_j \equiv_2 0 \text{ for } j = 1, \cdots, s) \wedge (r > 0 \vee t \equiv_2 1)$

A005843: The nonnegative even numbers: a(n) = 2n
$t > 0$

A004613: Numbers that are divisible only by primes congruent 1 mod 4.
$r > 0 \wedge t = s = 0$

A004614: Numbers that are divisible only by primes congruent 3 mod 4.
$s > 0 \wedge t = r = 0$

A018825: Numbers that are not the sum of 2 nonzero squares.
Complement of A000404.
$n \in A018825 \Leftrightarrow n \notin A000404 \Leftrightarrow (b_j \equiv_2 1 \text{ for } j = 1, \cdots, s) \vee (r = 0 \wedge t \equiv_2 0)$

A337140: Numbers n such that n is the sum of two positive integers a and b such that their product p = ab equals a perfect square.
$A005843 \subset A337140$. Proof $n \in A005843$ then $n = 2k = k + k$ and $p = k * k$ is a perfect square.
$A004613 \subset A337140$. Proof $n \in A004613$ then $n = p_i \cdot n_1 = (u^2 + v^2) \cdot n_1 = u^2 n_1 + v^2 n_1$ and $p = u^2 n_1 + v^2 n_1 = (uvn_1)^2$ is a perfect square.
$n \in A337140 \setminus (A005843 \cup A004613) \Leftrightarrow n = q_1^{b_1} * q_2^{b_2} * ... * q_s^{b_s} \in A018825$ Let now $n = p_i * n_1 = p_i * (u^2 + v) \Rightarrow p = p_i^2 u^2 v$ with v is not square for all $u^2 < n_1$.

3.3. **How many singulars for (c, s_a), if $s_a s_b$ is a perfect square?** .
The number of triangles in each skeleton is stated in table 2. It is remarkable that at constant c the number of singular solutions is equal for each possible $(s_a s_b)$.

3.4. **Are there Heron tirangles, that are singulars to 2 or 3 sides?** .
Table 1 lists the number of singular Heron skeletons listed in the column type. The triangle with sides (8,5,5) with type OI-3 is an obtused-angled isoscele triangle with three singular Heron skeletons. The triangle with sides (41,40,17) is acute-angled and has 1 singular Heron skeleton. It is remarkable that for primitive triangles with area up to 396:

- there are no triangles with two singular skeletons.
- there are only three triangles with three singular skeletons.
- Only triangle (8,5,5) has three singular skeletons with only one heron trilateral, each.

3.5. **How many series for (c,p), when p is not a perfect square?** .

Theorem 3.3 in Conrad's Pell's Equations II [4](See C.5 on page 8) is basic to this question.

3.6. **Some results with heroncx.** .

Heroncx(3,1):

r	a	b	c	F
1	5	4	3	6
$\frac{4}{3}$	26	25	3	36
$\frac{7}{5}$	149	148	3	210
$\frac{24}{17}$	866	865	3	1224
$\frac{41}{29}$	5045	5044	3	7134
$\frac{140}{99}$	29402	29401	3	41580
$\frac{239}{169}$	171365	171364	3	242346
..	..	..	..	..

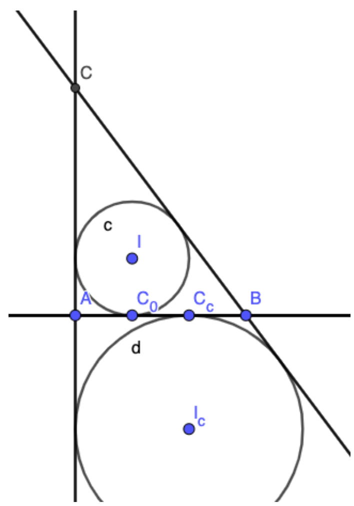

FIGURE 6. $\mathcal{H}(5,4,3)$

The first row is the rectangled Heron trilateral HT(5,4,3).
Some additional values are The second row is the obtuse-angled Heron trilateral HT(26,25,3).

s	s_a	s_b	s_c	F	r	r_a	r_b	r_c
6	1	2	3	6	1	6	3	2

Some additional values are

s	s_a	s_b	s_c	F	$\frac{4}{3}$	r_a	r_b	r_c
27	1	2	24	36	$\frac{4}{3}$	36	18	$\frac{3}{2}$

3.7. Some strong relations between r and convergents of $\sqrt{s_a s_b}$. .

In the next table are the series of inradii r and exradii r_c compared with the convergents $p_2(i)/q_2(i)$ from the continued fraction of $\sqrt{2}$.

i	1	2	3	4	5	6	7	$\cdots$
r	1	4/3	7/5	24/17	41/29	140/99	239/169	$\cdots$
r_c	2	3/2	10/7	17/12	58/41	99/70	338/239	$\cdots$
$p_2(i)$	1	3	7	17	41	99	239	$\cdots$
$q_2(i)$	1	2	5	12	29	70	169	$\cdots$

It is remarkable that the convergents of $\sqrt{2}$ are alternating the inradius and the exradius at the side c. It seems that

$$r_i = \frac{p_2(i)}{q_2(i)} \text{ if i is odd and } r_i = \frac{2 \cdot q_2(i)}{p_2(i)} \text{ if i is even} \tag{10}$$

Something like equation 11 seems to be valid in many cases. Let $\sqrt{2}$ be replaced by $\sqrt{d}$ en let $p_d(i)/q_d(i)$ be the i^{th} convergent from the continued fraction of $\sqrt{d}$. Then let

$$r_i = \frac{p_d(i)}{q_d(i)} \text{ if i is odd and } r_i = \frac{d \cdot q_d(i)}{p_d(i)} \text{ if i is even} \tag{11}$$

is true for $i = 1, 2, 3$ when $(c, p) = (c, d) = (c, s_a, s_b, d)$ as in the following table:

c	3	4	6	6	7	7	7	8	9	9	11	11	11	12	$\cdots$
s_a	1	1	1	2	1	2	3	3	1	4	1	3	5	1	$\cdots$
s_b	2	3	5	4	6	5	4	5	8	5	10	8	6	11	$\cdots$
d	2	3	5	8	6	10	12	15	8	20	10	24	30	11	$\cdots$

Much more of this has been collected in table 3. It is checked for $k = 1, \cdots 4$. For several series of solutions, only the basic series belonging to β is c_0 or c is stated. Convergents also occur irregularly in the 'higher' series (not listed in the table, but marked with nqo* or nqe*).

3.8. heroncx(50,7). .

$(c, s_a) = (50, 7)$ gives $d = p = 7 \cdot 43 = 301$ and $n = c_0^2 = 625$
bhaskara(301) gives $5883392537695^2 - 301 \cdot 339113108232^2 = 1$.
$u = 5883392537695 + 339113108232\sqrt{301}$
Then follows the estimated value of $ymax$ in $bhaskara_g(d, n) = bhaskara_g(301, 25^2)$
$ymax = \sqrt{n \cdot u}/\sqrt{301} = 4.94294378511965e6$
and bhaskara has to check up to 49429437.
And this takes too much time for my solution with sagemath.

References

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[7] Wiki, *Heronian triangle*, https://en.m.wikipedia.org/wiki/Heronian_triangle
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Appendix A. heroncx(c,x)

```
#A Version 2.0  Actual program consists of 4 seperate parts.
def heroncx(hc,hx):
    hp = hx*(hc - hx)
    if is_square(hp):
        if hc % 2 == 0:
            return sqe(hc,hx)
        return sqo(hc,hx)
    if hc % 2 == 0:
        return nqe(hc,hx)
    return nqo(hc,hx)
#B c:even, p:perfect square B^2 - A^2 = -c0^2 => s_c = A - c0 and F = p1*B
def sqe(c,s_a):
    sol = [['r','a','b','c','F']]
    s_b = c - s_a
    p = s_a * s_b
    c0 = c / 2
    p1 = sqrt(p)
#C d1 > -d2 > 0, d1*d2 = -c0^2, 2|(d1-d2)
    c0di = divisors(c0^2)
    for d1 in c0di:
        if d1 > c0:
            d2 = -c0^2/d1
            if (d1 - d2) % 2 == 0:
                B = (d1 + d2)/2
                A = (d1 - d2)/2
                s_c = A - c0
                s = A + c0
                F = p1 * B
                r = F / s
                a = s_b + s_c
                b = s_a + s_c
                sol.append([r,a,b,c,F])
    return sol
#D c:odd, p:perfect square B'^2 - A'^2 = c^2 => s_c = (A'-c)/2 and F = p1*B'/2
def sqo(c,s_a):
    sol = [['r','a','b','c','F']]
    s_b = c - s_a
    p = s_a * s_b
    p1 = sqrt(p)
#E d1 > -d2 > 0, d1*d2 = -c^2, 2|(d1-d2)
    cdi = divisors(c^2)
```

```
    for d1 in cdi:
        if d1 > c:
            d2 = -c^2/d1
            if (d1 - d2) % 2 == 0:
                B1 = (d1 + d2)/2
                A1 = (d1 - d2)/2
                s_c = (A1 - c)/2
                s = (A1 + c)/2
                F = p1 * B1/2
                r = F/s
                a = s_b + s_c
                b = s_a + s_c
                sol.append([r,a,b,c,F])
    return sol
#F x^2-dy^2 = 1 input d output (a,b) smallest solution in pos.integers
def bhaskara(d):
    if is_square(d): return 'none'
    k.<sqd> = QuadraticField(d)
    cfd = continued_fraction(sqd)
    i = 0
    a = 0
    b = 0
    while a^2 - d*b^2 != 1:
        i = i + 1
        cvd = cfd.convergent(i)
        a = numerator(cvd)
        b = denominator(cvd)
    return (a,b)
#G x^2-d*y^2=n input d,n output sol = list van lists of solutions (a+b*sqd) with b>0
def bhaskara_g(d,n):
    if is_square(d):
        return 'none'
    k.<sqdn> = QuadraticField(d)
    cfdn = continued_fraction(sqdn)
    ab = bhaskara(d)
    u = ab[0] + ab[1]*sqdn
    ymax = sqrt(n*u)/sqdn
    sol = [d,(ab[0],ab[1])]
    YY = 0
    while YY <= floor(ymax):
        XX = sqrt(d*YY^2 + n)
        if XX in ZZ:
            sol.append([XX,YY])
```

```
        YY = YY + 1
    return sol
#H 1.3 A^2 - p0 B^2 = c0^2 ==> s_c = A - c0 and F = p0 p1 B
def nqe(c,s_a):
#I Calculation of p, p0, p1, c0
    sol = [['r','a','b','c','F']]
    c0 = c/2
    s_b = c - s_a
    p = s_a * s_b
    p0 = squarefree_part(p)
    p1 = sqrt(p/p0)
#J quotients of solutions of bhaskara_g can be a bhaskara-unit.
    solu = bhaskara_g(p0,c0^2)
#K    print (solu)
    k.<sq> = QuadraticField(p0)
    u = solu[1][0]+solu[1][1]*sq
    solv = []
    for j in range(2,len(solu)):
        solv.append(solu[j][0]+solu[j][1]*sq)
        solB = [solv[0]]
#L
    j = 1
    while j < len(solv):
        kk = 0
        test = 1
        while kk < len(solB):
            soltest = solv[j]/solB[kk]
            if soltest[0] in ZZ and soltest[1] in ZZ:
                test = 0
            kk = kk + 1
        if test == 1: solB.append(solv[j])
        j = j + 1
#M hier is solB de
    lenB = len(solB)
    for i in range(lenB):
        testB = solB[i]
        if testB in ZZ: solB[i] = u * solB[i]
        if not testB in ZZ:
            while testB[1] > 0:
                testB = testB/u
        solB[i] = testB * u
#N
    for a in solB:
```

```
        so = [a*u^i for i in range(aantal)]
        for a1 in so:
            if a1[1] > 0:
                s_c = a1[0] - c/2
                s = a1[0] + c/2
                F = sqrt(s*s_a*s_b*s_c)
                r = F/s
                a = s_b + s_c
                b = s_a + s_c
                sol.append([r,a,b,c,F])
            if a1[1] < 0:
                s_c = -a1[0] - c/2
                s = -a1[0] + c/2
                F = sqrt(s*s_a*s_b*s_c)
                r = F/s
                a = s_b + s_c
                b = s_a + s_c
                sol.append([r,a,b,c,F])
        sol.append(['..','..','..','..','..'])
    return sol
#O 1.4 A^2 - p2 B^2 = c^2 ==> s_c = (A - c)/2 and F = p0 p1 B /2
def nqo(c,s_a):
    sol = [['r','a','b','c','F']]
#P Calculation of p, p0, p1
    s_b = c - s_a
    p = s_a * s_b
    p0 = squarefree_part(p)
    p2 = 4 * p0
    p1 = sqrt(p/p0)
#Q
    solu = bhaskara_g(p2,c^2)
#K    print (solu)
    k.<sq> = QuadraticField(p2)
    u = solu[1][0]+solu[1][1]*sq
    solv = []
    for j in range(2,len(solu)):
        solv.append(solu[j][0]+solu[j][1]*sq)
#R
    solB = [solv[0]]
    j = 1
    while j < len(solv):
        kk = 0
        test = 1
```

```
        while kk < len(solB):
            soltest = solv[j]/solB[kk]
            if soltest[0] in ZZ and soltest[1] in ZZ:
                test = 0
            kk = kk + 1
        if test == 1: solB.append(solv[j])
        j = j + 1
#S
    lenB = len(solB)
    for i in range(lenB):
        testB = solB[i]
        if testB in ZZ: solB[i] = u * solB[i]
        if not testB in ZZ:
            while testB[1] > 0:
                testB = testB/u
        solB[i] = testB * u
#T
    for a in solB:
        so = [a * u^i for i in range(aantal)]
        for a1 in so:
            if a1[1] > 0:
                s_c = (a1[0] - c)/2
                s = (a1[0] + c)/2
                F = sqrt(s*s_a*s_b*s_c)
                r = F/s
                a = s_b + s_c
                b = s_a + s_c
                sol.append([r,a,b,c,F])
            if a1[1] < 0:
                s_c = -(a1[0] - c)/2
                s = -(a1[0] + c)/2
                F = sqrt(s*s_a*s_b*s_c)
                r = F/s
                a = s_b + s_c
                b = s_a + s_c
                sol.append([r,a,b,c,F])
        sol.append(['..','..','..','..','..'])
    return sol
#U
aantal = var('aantal')
aantal = a
(table(heroncx(c,x)))
```

Appendix B. tables

Table 1. Primitive Heron triangles up to area (F) is 396

F	2s	c	b	a	r	type
6	12	5	4	3	1	R-0
12	16	6	5	5	$\frac{3}{2}$	AI-1
12	18	8	5	5	$\frac{4}{3}$	OI-3
24	32	15	13	4	$\frac{3}{2}$	O-1
30	30	13	12	5	2	R-0
36	36	17	10	9	2	O-1
36	54	26	25	3	$\frac{4}{3}$	O-0
42	42	20	15	7	2	O-0
60	36	13	13	10	$\frac{10}{3}$	AI-1
60	40	17	15	8	3	R-1
60	50	24	13	13	$\frac{12}{5}$	OI-1
60	60	29	25	6	2	O-0
66	44	20	13	11	3	O-0
72	64	30	29	5	$\frac{9}{4}$	O-0
84	42	15	14	13	4	A-0
84	48	21	17	10	$\frac{7}{2}$	O-0
84	56	25	24	7	3	R-0
84	72	35	29	8	$\frac{7}{3}$	O-1
90	54	25	17	12	$\frac{10}{3}$	O-0
90	108	53	51	4	$\frac{5}{3}$	O-0
114	76	37	20	19	3	O-0
120	50	17	17	16	$\frac{24}{5}$	AI-1
120	64	30	17	17	$\frac{15}{4}$	OI-1
120	80	39	25	16	3	O-0
126	54	21	20	13	$\frac{14}{3}$	A-0
126	84	41	28	15	3	O-0
126	108	52	51	5	$\frac{7}{3}$	O-0
132	66	30	25	11	4	O-0
156	78	37	26	15	4	O-0
156	104	51	40	13	3	O-0
168	64	25	25	14	$\frac{21}{4}$	AI-1
168	84	39	35	10	4	O-0
168	98	48	25	25	$\frac{12}{7}$	O-1
180	80	37	30	13	$\frac{9}{2}$	O-1
180	90	41	40	9	4	R-1
198	132	65	55	12	$\frac{9}{2}$	O-0
204	68	26	25	17	6	A-0
210	70	29	21	20	6	R-0
210	70	28	25	17	6	A-0
210	84	39	28	17	5	O-0
210	84	37	35	12	5	R-0
210	140	68	65	7	3	O-0
210	300	149	148	3	$\frac{7}{5}$	O-0
216	162	80	73	9	$\frac{8}{3}$	O-1
234	108	52	41	15	$\frac{54}{13}$	O-0
240	90	40	37	13	$\frac{28}{3}$	O-1
252	84	35	34	15	6	A-0
252	98	45	40	13	$\frac{36}{7}$	O-3
252	144	70	65	9	$\frac{7}{2}$	O-1
264	96	44	37	15	$\frac{11}{2}$	O-0
264	132	65	34	33	4	O-0
270	108	52	29	27	5	O-0
288	162	80	65	17	$\frac{32}{9}$	O-3
300	150	74	51	25	4	O-0
300	250	123	122	5	$\frac{12}{5}$	O-0
306	108	51	37	20	$\frac{17}{3}$	O-0
330	100	44	39	17	$\frac{33}{5}$	O-0
330	110	52	33	25	6	O-0
330	132	61	60	11	5	R-0
330	220	109	100	11	3	O-0
336	98	41	40	17	$\frac{24}{7}$	A-1
336	112	53	35	24	6	O-0
336	128	61	52	15	$\frac{21}{8}$	O-1
336	392	195	193	4	$\frac{12}{7}$	O-1
360	90	36	29	25	8	A-1
360	100	41	41	18	$\frac{36}{5}$	AI-1
360	162	80	41	41	$\frac{40}{9}$	OI-1
390	156	75	68	13	5	O-0
396	176	87	55	34	$\frac{9}{2}$	O-0
396	198	97	90	11	4	O-0
396	242	120	109	13	$\frac{18}{11}$	O-1

Table 2. $(c, a) - n$ with $c = a + b$ with $a * b$ is square $(a, b, c \in \mathbb{N})$ and n is the number of singular Heron trilaterals

$(2,1)-0$	$(36,18)-2$	$(65,1)-4$	$(88,44)-4$	$(113,49)-1$
$(4,2)-0$	$(37,1)-1$	$(65,13)-4$	$(89,25)-1$	$(114,57)-4$
$(5,1)-1$	$(38,19)-1$	$(65,16)-4$	$(90,9)-7$	$(115,23)-4$
$(6,3)-1$	$(39,12)-4$	$(65,20)-4$	$(90,18)-7$	$(116,16)-1$
$(8,4)-1$	$(40,4)-4$	$(66,33)-4$	$(90,45)-7$	$(116,18)-1$
$(10,1)-1$	$(40,8)-4$	$(68,4)-1$	$(91,28)-4$	$(116,58)-1$
$(10,2)-1$	$(40,20)-4$	$(68,18)-1$	$(92,46)-1$	$(117,36)-7$
$(10,5)-1$	$(41,16)-1$	$(68,34)-1$	$(94,47)-1$	$(118,59)-1$
$(12,6)-1$	$(42,21)-4$	$(70,7)-4$	$(95,19)-4$	$(119,7)-4$
$(13,4)-1$	$(44,22)-1$	$(70,14)-4$	$(96,48)-10$	$(120,12)-13$
$(14,7)-1$	$(45,9)-7$	$(70,35)-4$	$(97,16)-1$	$(120,24)-13$
$(15,3)-4$	$(46,23)-1$	$(72,36)-7$	$(98,49)-2$	$(120,60)-13$
$(16,8)-2$	$(48,24)-7$	$(73,9)-1$	$(100,2)-2$	$(122,1)-1$
$(17,1)-1$	$(50,1)-2$	$(74,2)-1$	$(100,10)-2$	$(122,50)-1$
$(18,9)-2$	$(50,5)-2$	$(74,25)-1$	$(100,20)-2$	$(122,61)-1$
$(20,2)-1$	$(50,10)-2$	$(74,37)-1$	$(100,36)-2$	$(123,48)-4$
$(20,4)-1$	$(50,18)-2$	$(75,15)-7$	$(100,50)-2$	$(124,62)-1$
$(20,10)-1$	$(50,25)-2$	$(75,27)-7$	$(101,1)-1$	$(125,4)-3$
$(22,11)-1$	$(51,3)-4$	$(76,38)-1$	$(102,6)-4$	$(125,25)-3$
$(24,12)-4$	$(52,2)-1$	$(78,3)-4$	$(102,27)-4$	$(125,45)-3$
$(25,5)-2$	$(52,16)-1$	$(78,24)-4$	$(102,51)-4$	$(126,63)-7$
$(25,9)-2$	$(52,26)-1$	$(78,39)-4$	$(104,4)-4$	$(128,64)-5$
$(26,1)-1$	$(53,4)-1$	$(80,8)-7$	$(104,32)-4$	$(130,2)-4$
$(26,8)-1$	$(54,27)-3$	$(80,16)-7$	$(104,52)-4$	$(130,5)-4$
$(26,13)-1$	$(55,11)-4$	$(80,40)-7$	$(105,21)-13$	$(130,9)-4$
$(28,14)-1$	$(56,28)-4$	$(82,1)-1$	$(106,8)-1$	$(130,13)-4$
$(29,4)-1$	$(58,8)-1$	$(82,32)-1$	$(106,25)-1$	$(130,26)-4$
$(30,3)-4$	$(58,9)-1$	$(82,41)-1$	$(106,53)-1$	$(130,32)-4$
$(30,6)-4$	$(58,29)-1$	$(84,42)-4$	$(108,54)-3$	$(130,40)-4$
$(30,15)-4$	$(60,6)-4$	$(85,4)-4$	$(109,9)-1$	$(130,49)-4$
$(32,16)-3$	$(60,12)-4$	$(85,5)-4$	$(110,11)-4$	$(130,65)-4$
$(34,2)-1$	$(60,30)-4$	$(85,17)-4$	$(110,22)-4$	$(132,66)-4$
$(34,9)-1$	$(61,25)-1$	$(85,36)-4$	$(110,55)-4$	$(134,67)-1$
$(34,17)-1$	$(62,31)-1$	$(86,43)-1$	$(111,3)-4$	$(135,27)-10$
$(35,7)-4$	$(64,32)-4$	$(87,12)-12$	$(112,56)-7$	$(136,8)-4$

TABLE 3. $nqx - c - s_a - s_b - p - r_i$

nqx	c	s_a	s_b	d	r_i
nqo	3	1	2	2	$r_{2k-1} = p_2(2k-1)/q_2(2k-1), r_{2k} = 2 \cdot q_2(2k)/p_2(2k)$
nqe	4	1	3	3	$r_{2k-1} = p_3(2k-1)/q_3(2k-1), r_{2k} = 3 \cdot q_3(2k)/p_3(2k)$
nqo*	5	2	3	6	$r_{2k-1} = p_6(2k-1)/q_6(2k-1), r_{2k} = 6 \cdot q_6(2k)/p_6(2k)$
nqe	6	1	5	5	$r_{2k-1} = p_5(2k-1)/q_5(2k-1), r_{2k} = 5 \cdot q_5(2k)/p_5(2k)$
nqe	6	2	4	8	$r_{2k-1} = p_8(2k-1)/q_8(2k-1), r_{2k} = 8 \cdot q_8(2k)/p_8(2k)$
nqe	6	2	4	8	$r_{2k-1} = 2 \cdot p_2(2k-1)/q_2(2k-1), r_{2k} = 2 \cdot 2 \cdot q_2(2k)/p_2(2k)$
nqo	7	1	6	6	$r_{2k-1} = p_6(2k-1)/q_6(2k-1), r_{2k} = 6 \cdot q_6(2k)/p_6(2k)$
nqo	7	3	4	12	$r_{2k-1} = p_{12}(2k-1)/q_{12}(2k-1), r_{2k} = 12 \cdot q_{12}(2k)/p_{12}(2k)$
nqo	7	3	4	12	$r_{2k-1} = 2 \cdot p_3(2k)/q_3(2k), r_{2k} = 2 \cdot 3 \cdot q_3(4k)/p_3(4k)$
nqo	7	2	5	10	$r_{2k-1} = p_{10}(2k-1)/q_{10}(2k-1), r_{2k} = 10 \cdot q_{10}(2k)/p_{10}(2k)$
nqe	8	1	7	7	$r_i = 7 \cdot q_7(2i)/p_7(2i)$
nqe	8	2	6	12	$r_{2k-1} = 2 \cdot p_3(2k-1)/q_3(2k-1), r_{2k} = 2 \cdot 3 \cdot q_3(2k)/p_3(2k)$
nqe	8	3	5	15	$r_{2k-1} = p_{15}(2k-1)/q_{15}(2k-1), r_{2k} = 15 \cdot q_{15}(2k)/p_{15}(2k)$
nqo	9	1	8	8	$r_{2k-1} = p_8(2k-1)/q_8(2k-1), r_{2k} = 8 \cdot q_8(2k)/p_8(2k)$
nqo	9	1	8	8	$r_{2k-1} = 2 \cdot p_2(2k-1)/q_2(2k-1), r_{2k} = 2 \cdot 2 \cdot q_2(2k)/p_2(2k)$
nqo	9	2	7	14	$r_i = 14 \cdot q_{14}(2i)/p_{14}(2i)$
nqo	9	3	6	18	$r_{2k-1} = 3 \cdot p_2(2k-1)/q_2(2k-1), r_{2k} = 3 \cdot 2 \cdot q_2(2k)/p_2(2k)$
nqo	9	4	5	20	$r_{2k-1} = p_{20}(2k-1)/q_{20}(2k-1), r_{2k} = 20 \cdot q_{20}(2k)/p_{20}(2k)$
nqo	9	4	5	20	$r_{2k-1} = 2 \cdot p_5(2k-1)/q_5(2k-1), r_{2k} = 2 \cdot 5 \cdot q_5(2k)/p_5(2k)$
nqe*	10	3	7	21	$r_{2k-1} = p_{21}(6k-3)/q_{21}(6k-3), r_{2k} = 21 \cdot q_{21}(6k)/p_{21}(6k)$
nqe	10	4	6	24	$r_{2k-1} = p_{24}(2k-1)/q_{24}(2k-1), r_{2k} = 24 \cdot q_{24}(2k)/p_{24}(2k)$
nqe	10	4	6	24	$r_{2k-1} = 2 \cdot p_6(2k-1)/q_6(2k-1), r_{2k} = 2 \cdot 6 \cdot q_6(2k)/p_6(2k)$
nqo	11	1	10	10	$r_{2k-1} = p_{10}(2k-1)/q_{10}(2k-1), r_{2k} = 10 \cdot q_{10}(2k)/p_{10}(2k)$
nqo	11	2	9	18	$r_{2k-1} = 3 \cdot p_2(2k-1)/q_2(2k-1), r_{2k} = 3 \cdot 2 \cdot q_2(2k)/p_2(2k)$
nqo	11	3	8	24	$r_{2k-1} = p_{24}(2k-1)/q_{24}(2k-1), r_{2k} = 24 \cdot q_{24}(2k)/p_{24}(2k)$
nqo	11	3	8	24	$r_{2k-1} = 2 \cdot p_6(2k-1)/q_6(2k-1), r_{2k} = 2 \cdot 6 \cdot q_6(2k)/p_6(2k)$
nqo	11	4	7	28	$r_i = 28 \cdot 28 q_{28}(2i)/p_{28}(2i)$
nqo	11	4	7	28	$r_i = 2 \cdot 7 \cdot q_7(4i)/p_7(4i)$
nqo	11	5	6	30	$r_{2k-1} = p_{30}(2k-1)/q_{30}(2k-1), r_{2k} = 30 \cdot q_{30}(2k)/p_{30}(2k)$
nqo	12	1	11	11	$r_{2k-1} = p_{11}(2k-1)/q_{11}(2k-1), r_{2k} = 11 \cdot q_{11}(2k)/p_{11}(2k)$
nqe*	12	2	10	20	$r_{2k-1} = p_{20}(2k-1)/q_{20}(2k-1), r_{2k} = 20 \cdot q_{20}(2k)/p_{20}(2k)$
nqe*	12	2	10	20	$r_{2k-1} = 2 \cdot p_5(2k-1)/q_5(2k-1), r_{2k} = 2 \cdot 5 \cdot q_5(2k)/p_5(2k)$
nqe	12	3	9	27	$r_{2k-1} = p_{27}(2k-1)/q_{27}(2k-1), r_{2k} = 27 \cdot q_{27}(2k)/p_{27}(2k)$
nqe	12	3	9	27	$r_{2k-1} = 3 \cdot p_3(6k-3)/q_3(6k-3), r_{2k} = 3 \cdot 3 \cdot q_3(6k)/p_3(6k)$
nqe	12	4	8	32	$r_{2k-1} = ??$
nqe	12	4	8	32	$r_{2k} = 32 \cdot q_{32}(2k)/p_{32}(2k) = 2 \cdot 8 \cdot q_8(2k)/p_8(2k)) = 4 \cdot 2 \cdot q_2(2k)/p_2(2k))$
nqo	12	5	7	35	$r_{2k-1} = p_{35}(2k-1)/q_{35}(2k-1), r_{2k} = 35 \cdot q_{35}(2k)/p_{35}(2k)$